New perspective on water-surface profiles using critical slope

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ABSTRACT

A new perspective on water-surface profiles is made possible by expressing the gradually varied flow equation in terms of the critical slope S_c. In this way, the flow-depth gradient (dy/dx) is shown to be strictly limited to values *outside* the range encompassed by S_c and S_o, in which S_o is the bed slope. This new perspective improves and completes the definition of flow-depth-gradient ranges in the analysis of water surface profiles.

INTRODUCTION

Computations of gradually varied flow (GVF) are part of the routine practice of hydraulic engineering. The GVF equation describes steady gradually varied flow in open channels (Chow 1959; Henderson 1966). The conventional GVF equation is expressed in terms of bed slope S_o, friction slope S_f, and Froude number **F**. In this paper, the GVF equation is alternately expressed in terms of bed slope S_o, critical slope S_c, and Froude number **F**. Analysis of this equation reveals that the flow-depth gradient d*y*/d*x* is strictly limited to values outside the range encompassed by S_c and S_o. This improves and completes the definition of flow-depth-gradient ranges in the analysis of water surface profiles.

GRADUALLY VARIED FLOW EQUATION

The GVF equation is (Chow 1959, p. 220; Henderson 1966, p. 130):

$$\frac{dy}{dx} = \frac{S_0 - S_f}{1 - \left[\left(Q^2 T \right) / \left(g\ A^3 \right) \right]} \tag{1}$$

in which y = flow depth, x = distance along the channel, d*y*/d*x* = flow-depth gradient, Q = discharge, T = top width, A = flow area, and g = gravitational acceleration. This equation is valid for small bed slopes, which is usually the case.

The friction slope in terms of the Chezy coefficient C is (Chow 1959):

$$S_f = \frac{Q^2}{C^2 A^2 R} \tag{2}$$

in which $R = A/P$ = hydraulic radius, and P = wetted perimeter.

The Froude number in terms of discharge is (Chow 1959):

$$F^2 = \frac{Q^2 T}{g A^3} \tag{3}$$

Combining Equations 2 and 3 leads to:

$$S_f = \left(\frac{P}{T}\right)\left(\frac{g}{C^2}\right)F^2 \qquad (4)$$

At normal critical flow, **F** = 1, and the critical slope is, from Equation 4,

$$S_c = \left(\frac{P_c}{T_c}\right)\left(\frac{g}{C^2}\right) \qquad (5)$$

Combining Equations 1, 4, and 5:

$$\frac{dy}{dx} = \frac{S_0 - S_c F^2}{1 - F^2} \qquad (6)$$

which is strictly valid only as *P* approaches *T*, i.e., for a hydraulically wide channel. Then, Equation 6 is an asymptotic solution of steady-gradually-varied flow for hydraulically wide channels.

For ease of expression, we rename the flow-depth gradient $S_y = dy/dx$, and solve for Froude number from Equation 6:

$$F^2 = \frac{S_0 - S_y}{S_c - S_y} \qquad (7)$$

Since $\mathbf{F}^2 \geq 0$, the flow-depth gradient *must* satisfy the following inequalities:

$$\boldsymbol{S_0 \geq S_y \leq S_c} \qquad (8)$$

$$\boldsymbol{S_0 \leq S_y \geq S_c} \tag{9}$$

which effectively limits the flow-depth gradient to values outside the range encompassed by S_o and S_c.

Furthermore, Equation 6 can be alternately expressed as follows:

$$\frac{S_y}{S_c} = \frac{(S_0/S_c) - F^2}{1 - F^2} \tag{10}$$

Equation 10 is the GVF equation in terms of bed slope S_o , critical slope S_c , and Froude number **F**. The bed slope could be positive (steep, critical, or mild), zero (horizontal), or negative (adverse). The critical slope (Equation 5) and Froude number squared (Equation 3) are *always* positive.

CLASSIFICATION OF WATER SURFACE PROFILES

We use Equation 10 to develop a classification of water surface profiles based solely on the three dimensionless parameters: S_y/S_c , S_o/S_c , and **F**. For the sake of completeness, subcritical flow is defined as that for which the flow depth is greater than the critical depth ($\mathbf{F}^2 < 1$) (Chow 1959; Henderson 1966). Paralleling this widely accepted definition, *subnormal* flow is defined as that for which the flow depth is *greater* than the normal depth [$\mathbf{F}^2 < S_o/S_c$]. *Supernormal* flow is defined as that for which the flow depth is *smaller* than the normal depth [$\mathbf{F}^2 > S_o/S_c$] (USDA SCS 1971).

Using Eq. 10, the following combinations of GVF profiles are possible:

- TYPE 1: SUBCRITICAL/SUBNORMAL
  - -- Steep: S_1
  - -- Critical: C_1
  - -- Mild: M_1
- TYPE 2A: SUPERCRITICAL/SUBNORMAL
  - -- Steep: S_2

- TYPE 2B: SUBCRITICAL/SUPERNORMAL

  -- Mild: M_2

  -- Horizontal: H_2

  -- Adverse: A_2

- TYPE 3: SUPERCRITICAL/SUPERNORMAL

  -- Steep: S_3

  -- Critical: C_3

  -- Mild: M_3

  -- Horizontal: H_3

  -- Adverse: A_3

A summary of the twelve possible water surfaces profiles is shown in Table 1. The classification follows directly from the governing equation (Equation 10) shown at the top of the table. It is seen that the general type of profile (Type 1, 2, or 3) determines the sign of S_y/S_c (Column 2) and thus, the classification of either backwater or drawdown (Column 3). Also, the general type of profile determines the feasible range of S_o/S_c (Column 4) and thus, the existence of specific profiles types (Steep, Critical, Mild, Horizontal, or Adverse) within each general type. Note that not all combinations of S_y/S_c and S_o/S_c are feasible.

Unlike the description available in standard references (Chow 1959; Henderson 1966), the flow-depth-gradient ranges (Table 1, Columns 7 and 8) are now complete for all twelve water surfaces profiles. Significantly, the flow depth gradient S_y is shown to be outside the range encompassed by S_c and S_o.

Figure 1 shows a graphical representation of flow-depth-gradient ranges in the water surface profiles. The arrow shows the direction of computation. For instance, the depth gradient for the S_3 profile (supercritical/supernormal) decreases from S_c (a finite positive value) to 0 (asymptotic to normal flow). Likewise, the depth gradient for the C_1 (subcritical/subnormal) and C_3 (supercritical/supernormal) profiles is constant and equal to $S_o = S_c$.

SUMMARY

The gradually varied flow equation is alternatively expressed in terms of the critical slope S_c. In this way, the flow dept gradient (dy/dx) is shown to be strictly limited to values outside the range encompassed by S_c and S_o. This new perspective completes the definition of depth-gradient ranges for all water surface profiles. For instance, the flow-depth gradient for the S_3 profile decreases from S_c (a finite positive value) to 0 (asymptotic to normal depth). Likewise, the flow depth gradient for the C_1 and C_3 profiles is constant and equal to $S_o = S_c$.

APPENDIX I. REFERENCES

Chow, V. T. (1959). *Open-channel hydraulics*. McGraw-Hill, New York.

Henderson, F. M. (1966). *Open channel flow*. MacMillan, New York.

USDA Soil Conservation Service (1971). Classification system for varied flow in prismatic channels. *Technical Release No. 47* (TR-47), Washington, D.C.

APPENDIX II. NOTATION

The following symbols are used in this paper:

A = flow area;

C = Chezy coefficient;

$\boldsymbol{F}$ = Froude number;

g = gravitational acceleration;

P = wetted perimeter;

Q = discharge;

R = hydraulic radius;

S_c = critical slope;

S_f = friction slope;

S_o = bed slope;

S_y = flow-depth gradient;

T = top width;

x = distance along the channel; and

y = flow depth.

Table 1. Classification of Water Surface Profiles.

Governing Equation: $S_y/ S_c = [(S_0/ S_c)- \mathbf{F}^2]/(1- \mathbf{F}^2)$								
No. (1)	S_y/ S_c (2)	Profile (3)	S_o/ S_c (4)	Slope (5)	Depth relations (6)	S_y varies From (7)	To (8)	Type (9)
1. SUBCRITICAL/SUBNORMAL FLOW[1]: $1>\mathbf{F}^2< S_0/ S_c$								
1	Positive	Backwater	> 1	Steep	$y > y_c > y_n$	S_o	∞	S_1
2	Positive	Backwater	= 1	Critical	$y > y_c = y_n$	$S_o= S_c$	$S_o = S_c$	C_1
3	Positive	Backwater	>0; <1	Mild	$y > y_n > y_c$	S_o	0	M_1
2A. SUPERCRITICAL/SUBNORMAL FLOW[2]: $1< \mathbf{F}^2< S_0/ S_c$								
4	Negative	Drawdown	>1	Steep	$y_c > y > y_n$	$-\infty$	0	S_2
2B. SUBCRITICAL/SUPERNORMAL FLOW[3]: $1> \mathbf{F}^2> S_o/ S_c$								
5	Negative	Drawdown	> 0; < 1	Mild	$y_n > y > y_c$	$-\infty$	0	M_2
6	Negative	Drawdown	= 0	Horizontal	$y> y_c ; y_n \to \infty$	$-\infty$	$S_o = 0$	H_2
7	Negative	Drawdown	< 0	Adverse	$y> y_c ; y_n\to \infty$	$-\infty$	$S_o < 0$	A_2
3. SUPERCRITICAL/SUPERNORMAL FLOW[4]: $1< \mathbf{F}^2> S_0/ S_c$								
8	Positive	Backwater	>1	Steep	$y_c > y_n > y$	S_c	0	S_3
9	Positive	Backwater	= 1	Critical	$y_c = y_n > y$	$S_o = S_c$	$S_o = S_c$	C_3
10	Positive	Backwater	> 0; < 1	Mild	$y_n > y_c > y$	S_c	∞	M_3
11	Positive	Backwater	= 0	Horizontal	$y_c > y; y_n \to \infty$	S_c	∞	H_3
12	Positive	Backwater	< 0	Adverse	$y_c > y; y_n \to \infty$	S_c	∞	A_3

[1]Given that $S_0/Sc > F^2 >0$, no horizontal or adverse profiles are possible in subcritical /subnormal flow.
[2]Given that $S_0/Sc >1$, no critical, mild, horizontal or adverse profiles are possible in supercritical/subnormal flow.
[3]Given that $S_0/Sc <1$, no steep or critical profiles are possible in subcritical/supernormal flow.
[4]Given that S_0/Sc is not limited, all five profiles are possible in supercritical/supernormal flow

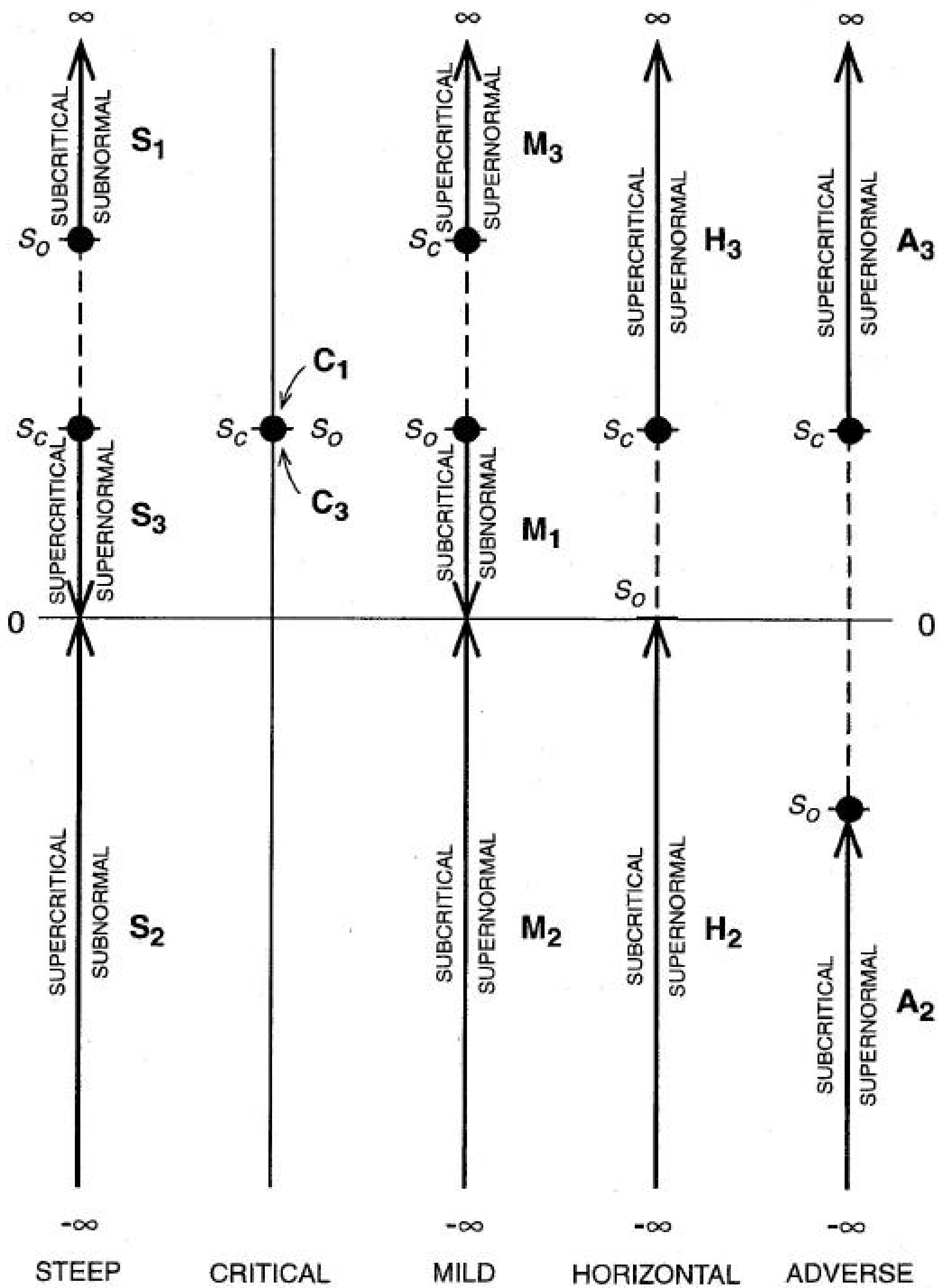

Fig. 1 Graphical representation of flow-depth-gradient ranges in water surface profiles.